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ON UNIQUENESS OF THE CANONICAL TENSOR DECOMPOSITION WITH SOME FORM OF SYMMETRY*

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Abstract. We study the uniqueness of the decomposition of an nth order tensor (also called n-way array) into a sum of R rank-1 terms (where each term is the outer product of n vectors). This decomposition is also known as Parafac or Candecomp, and a general uniqueness condition for $n = 3$ was obtained by Kruskal in 1977 [*Linear Algebra Appl.*, 18 (1977), pp. 95–138]. More recently, Kruskal's uniqueness condition has been generalized to $n \geq 3$, and less restrictive uniqueness conditions have been obtained for the case where the vectors of the rank-1 terms are linearly independent in (at least) one of the n modes. We consider the decomposition with some form of symmetry, and prove necessary, sufficient, and necessary and sufficient uniqueness conditions analogous to the asymmetric case. For $n = 3$, 4, 5, we also prove generic uniqueness bounds on R. Most of these conditions are easy to check. Throughout, we emphasize the analogies and striking differences between the symmetric and asymmetric cases.

Key words. tensor decomposition, uniqueness, symmetry, Parafac, Candecomp, Indscal

AMS subject classifications. 15A18, 15A69

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1. Introduction.

Tensors of order n are defined on the outer product of n linear spaces, $\mathcal{T}_\ell$, $1 \leq \ell \leq n$. Once bases of spaces $\mathcal{T}_\ell$ are fixed, they can be represented by n-way arrays. For simplicity, tensors are usually assimilated with their array representation.

We consider the nth order tensor decomposition of the form

$$\underline{\mathbf{X}} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \circ \mathbf{a}_r^{(2)} \circ \cdots \circ \mathbf{a}_r^{(n)}, \tag{1.1}$$

where $\underline{\mathbf{X}} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_n}$ is an nth order tensor (or n-way array), $\mathbf{a}_r^{(j)} \in \mathbb{R}^{I_j}$ are vectors, and $\circ$ denotes the outer vector product. For vectors $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}$, the outer vector product $\mathbf{a}^{(1)} \circ \cdots \circ \mathbf{a}^{(n)}$ is an nth order tensor with entries $a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_n}^{(n)}$. We refer to $\underline{\mathbf{X}}$ in (1.1) as having n *modes*, and the j in $\mathbf{a}_r^{(j)}$ corresponds to mode j. Let $\mathbf{A}^{(j)} = [\mathbf{a}_1^{(j)} | \mathbf{a}_2^{(j)} | \ldots | \mathbf{a}_R^{(j)}]$ denote the jth component matrix. Hence, matrix $\mathbf{A}^{(j)}$ has size $I_j \times R$. We denote an nth order decomposition (1.1) as $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$. Note that when the modes of $\underline{\mathbf{X}}$ are permuted in (1.1), the component matrices $\mathbf{A}^{(j)}$ are permuted identically.

An nth order tensor has rank 1 if it can be written as the outer product of n vectors. The rank of an nth order tensor $\underline{\mathbf{X}}$ is defined as the smallest number of rank-1 tensors whose sum equals $\underline{\mathbf{X}}$. Hence, (1.1) decomposes $\underline{\mathbf{X}}$ into R rank-1 terms. Hitchcock [15], [16] introduced tensor rank and the related tensor decomposition (1.1). The same decomposition was proposed independently by Carroll and Chang [3] and Harshman [14]

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for component analysis of nth order tensors. They named it Candecomp and Parafac, respectively.

For a given nth order tensor and number R of rank-1 components, a best fitting decomposition (1.1) is usually found by an iterative algorithm. The most well-known algorithm is alternating least squares. A comparison of algorithms for $n = 3$ can be found in Tomasi and Bro [46]. Note that a best fitting decomposition is a best rank-R approximation of the tensor.

Real-valued applications of tensor decompositions occur in psychology and chemistry; see Kroonenberg [22], Kiers and Van Mechelen [19], and Smilde, Bro, and Geladi [32]. Complex-valued tensor decompositions are used in, e.g., signal processing and telecommunications research; see Sidiropoulos, Giannakis, and Bro [30], Sidiropoulos, Bro, and Giannakis [31], and De Lathauwer and Castaing [8]. For a general overview of applications of the decomposition (1.1) and related decompositions, see Kolda and Bader [20] or Acar and Yener [1].

A drawback of computing a best fitting tensor decomposition (1.1) is that an optimal solution may not exist. Indeed, a tensor may not have a best rank-R approximation. This is due to the fact that the set of tensors of rank at most R is not closed for $R \geq 2$; see De Silva and Lim [11]. In such cases, some columns of the $\mathbf{A}^{(j)}$ become nearly linearly dependent and large in magnitude while running an iterative algorithm designed to find a best rank-R approximation; see Krijnen, Dijkstra, and Stegeman [21]. This phenomenon is known as "diverging components" or "degeneracy"; see Kruskal, Harshman, and Lundy [24] and Stegeman [33], [34], [35], [36]. This problem can be fixed by including interaction terms in the decomposition; see Stegeman and De Lathauwer [42] for the case $n = 3$ and $I_3 = 2$, Rocci and Giordani [28] for the case $n = 3$ and $R = 2$, and Stegeman [39] for a general approach for $n = 3$ and $R \leq \min(I_1, I_2, I_3)$.

An attractive feature of the decomposition (1.1) is that it is unique up to permutation and scaling under mild conditions. We define the uniqueness of (1.1) as follows.

DEFINITION 1.1. *The decomposition* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *is called* unique up to permutation and scaling *if any alternative decomposition* $(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)})$ *satisfies* $\mathbf{B}^{(j)} = \mathbf{A}^{(j)}\mathbf{\Pi}\mathbf{\Lambda}_j$, $j = 1, \ldots, n$, *with* $\mathbf{\Pi}$ *an* $R \times R$ *permutation matrix and* $\mathbf{\Lambda}_j$ *nonsingular diagonal matrices such that* $\prod_{j=1}^{n} \mathbf{\Lambda}_j = \mathbf{I}_R$. ∎

Hence, an nth order decomposition is unique up to permutation and scaling if the only ambiguities it contains are the permutation of the R rank-1 components and the scaling of the n vectors constituting each rank-1 component. Two decompositions that are equal up to these indeterminacies are called *equivalent*.

The classical uniqueness condition for $n = 3$ is due to Kruskal [23]. Kruskal's condition relies on a particular concept of matrix rank that he introduced, which has been named k-rank (after him). Specifically, the k-rank of a matrix is the largest number x such that every subset of x columns of the matrix is linearly independent. We denote the k-rank of a matrix $\mathbf{A}$ as $k_{\mathbf{A}}$. For a decomposition $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)})$, Kruskal [23] proved that

$$2R + 2 \leq k_{\mathbf{A}^{(1)}} + k_{\mathbf{A}^{(2)}} + k_{\mathbf{A}^{(3)}} \tag{1.2}$$

is a sufficient condition for uniqueness up to permutation and scaling. A more condensed and accessible proof of (1.2) was given by Stegeman and Sidiropoulos [41]. See Rhodes [27] for a different approach. Kruskal's uniqueness condition was generalized to $n \geq 3$ by Sidiropoulos and Bro [29]: for a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ the uniqueness condition becomes

$$2R + (n-1) \leq \sum_{j=1}^{n} k_{\mathbf{A}^{(j)}}. \tag{1.3}$$

From (1.3), it can be seen that the uniqueness condition becomes less restrictive as the order n increases. Indeed, when increasing n by one the right-hand side of (1.3) increases with an additonal k-rank, while the left-hand side increases by one only. Note that the k-ranks in (1.3) may not be zero. Indeed, $k_{\mathbf{A}^{(j)}} = 0$ implies (by convention) that $\mathbf{A}^{(j)}$ has an all-zero column and, hence, that the decomposition contains an all-zero term among its R terms. In this case we have nonuniqueness: an alternative decomposition into $R-1$ rank-1 terms is possible.

Less restrictive uniqueness conditions have been obtained for the case where (at least) one of the component matrices $\mathbf{A}^{(j)}$ has rank R; i.e., the vectors $\mathbf{a}_r^{(j)}$, $r = 1, \ldots, R$, are linearly independent in (at least) one mode j. See Jiang and Sidiropoulos [18] for $n = 3$, De Lathauwer [7] for $n = 3$ and $n = 4$, and Stegeman [38] for $n \geq 3$.

In this paper, we consider the nth order decomposition (1.1) with some form of symmetry, that is, a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ in which some of the component matrices are identical. For example, if $n = 3$ and $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$, then the entries of $\underline{\mathbf{X}}$ are symmetric in the first two modes: $x_{ijk} = x_{jik}$ for all i, j, k. If $n = 4$ and $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$, then we have $x_{ijkl} = x_{jikl}$ for all i, j, k, l. We assume that the modes of the decomposition are permuted such that identical component matrices occur in the first few modes. In what follows, we will sometimes refer to a decomposition with some form of symmetry as a "symmetric decomposition."

Due to the scaling indeterminacy, a symmetric decomposition can have component matrices with proportional columns (e.g., $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}\mathbf{\Lambda}$ with a nonsingular diagonal matrix $\mathbf{\Lambda}$) instead of identical component matrices. When (at least) one mode is excluded from the symmetry, the constants of proportionality can be absorbed in the component matrix corresponding to that mode. For convenience, we assume identical component matrices instead of proportional columns.

Applications of the tensor decomposition (1.1) with some form of symmetry are the following. The case $n = 3$ and $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$ corresponds to the Indscal model introduced by Carrol and Chang [3]. Indscal is a multidimensional scaling method for the case where several symmetric matrices of proximities or (dis)similarities are available for the same objects. In signal processing, the same form of symmetry occurs in the so-called second order blind identification (SOBI) method; see Belouchrani et al. [2]. Here, the goal is to separate signal sources from an observed mixture of signals by decomposing a set of covariance matrices, each measured at a different point in time. For the underdetermined case of SOBI we refer to De Lathauwer and Castaing [10]. Other blind separation methods resulting in a decomposition with the same form of symmetry include Pham and Cardoso [26], who also use covariance matrices, and Yeredor [47], who uses second order derivatives of the characteristic function of the observed signal mixture. For an overview of these models, we refer to Yeredor [48]. For $n = 4$ and $\mathbf{A}^{(j)} = \mathbf{A}$, $j = 1, 2, 3, 4$, the tensor decomposition (1.1) describes the basic structure of fourth order cumulants of multivariate data on which a lot of algebraic methods for independent component analysis (ICA) are based (Comon [4], De Lathauwer, De Moor, and Vandewalle [6], and Hyvärinen, Karhunen, and Oja [17]). For an ICA algorithm explicitly using (1.1) with this form of symmetry we refer to De Lathauwer, Castaing, and Cardoso [9]. For $n = 5$, $\mathbf{A}^{(1)} = \mathbf{A}^{(3)}$, and $\mathbf{A}^{(2)} = \mathbf{A}^{(4)}$, the decomposition (1.1) can be found in De Vos et al. [12]. This form of decomposition results from combining the third order decomposition with

ICA in one mode. Finally, the case $n = 5$ and $\mathbf{A}^{(j)} = \mathbf{A}$, $j = 1, 2, 3, 4$, appears in Ferréol, Albera, and Chevalier [13] where a blind separation method is proposed that uses a set of fourth order cumulants, each measured at a different point in time.

In signal processing applications, forms of symmetry may occur with, e.g., $\mathbf{A}^{(1)}$ equal to the complex conjugate of $\mathbf{A}^{(2)}$. The description of the applications above refers to the real case (if a complex case exists also), and throughout we will consider real-valued decompositions. However, our results can be translated easily to the complex case. This will be elaborated upon in the discussion section at the end of this paper.

We focus on the uniqueness properties of (1.1) when some form of symmetry is present. Uniqueness of such a decomposition is not necessarily identical to uniqueness of its asymmetric counterpart. Indeed, if a particular form of symmetry is inherent to the decomposition, then this form of symmetry must also be present in an alternative decomposition. Hence, the set of symmetric alternative decompositions is a subset of the set of all alternative decompositions. However, some uniqueness conditions for the asymmetric case can still be used. If the uniqueness condition (1.3) holds for a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ with some form of symmetry, then the decomposition is unique up to permutation and scaling. Hence, there are no nonequivalent asymmetric or symmetric alternatives. One of the main results of this paper is that if (at least) one of the component matrices $\mathbf{A}^{(j)}$ has rank R, and mode j is excluded from the symmetry, then uniqueness with respect to the set of symmetric alternatives is identical to uniqueness with respect to the set of asymmetric alternatives.

In Stegeman [38] an overview is presented of necessary, sufficient, necessary and sufficient, and generic uniqueness conditions for the asymmetric nth order decomposition (1.1). The generic uniqueness conditions hold for decompositions with generic $\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)}$ and $\operatorname{rank}(\mathbf{A}^{(n)}) = R$ and give a bound on R in terms of $I_1, \ldots, I_{n-1}$. In this paper, we prove symmetric analogues of most of these conditions. Although the symmetric uniqueness conditions are mostly analogous to the asymmetric ones, sometimes a more complicated proof is needed when symmetry is present. The most striking difference concerns the generic uniqueness bounds on R, which are much more restrictive in the presence of symmetry.

Our analysis yields more insight into the uniqueness of (1.1) with some form of symmetry. Moreover, our results include easy-to-check uniqueness conditions, and they can be applied to an important class of applications. The organization of this paper is as follows. Section 2 contains definitions and notation. In section 3, we prove our necessary uniqueness conditions. In section 4, we consider the case where (at least) one of the component matrices $\mathbf{A}^{(j)}$ has rank R, and mode j is excluded from the symmetry. For convenience, we take $j = n$. For decompositions with symmetry, we prove necessary and sufficient uniqueness conditions and an easy-to-check sufficient uniqueness condition analogous to [38]. Moreover, we show that all alternative decompositions have the same form of symmetry as the original decomposition when $\operatorname{rank}(\mathbf{A}^{(n)}) = R$ and $\operatorname{rank}(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)}) = R$. In section 5, we prove generic uniqueness bounds for $n = 3$, 4, 5 and several forms of symmetry. Each of sections 3, 4, and 5 starts with a summary of the uniqueness conditions proven in [38] for the asymmetric case. Section 6 contains several examples illustrating our results from sections 4 and 5 for the case $n = 3$ and $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$. Finally, section 7 contains a discussion of our results.

2. Definitions and notation. We will denote vectors as $\mathbf{x}$, matrices (second order tensors, 2-way arrays) as $\mathbf{X}$, and higher order tensors (multiway arrays) as $\underline{\mathbf{X}}$. We use $\otimes$ to denote the usual Kronecker product, and $\odot$ denotes the (columnwise) Khatri–Rao product; i.e., for matrices $\mathbf{X}$ and $\mathbf{Y}$ with R columns, $\mathbf{X} \odot \mathbf{Y} = [\mathbf{x}_1 \otimes \mathbf{y}_1 | \ldots | \mathbf{x}_R \otimes \mathbf{y}_R]$.

The transpose of $\mathbf{X}$ is denoted as $\mathbf{X}^T$, and $\text{diag}(\mathbf{x})$ denotes the diagonal matrix with the entries in vector $\mathbf{x}$ on its diagonal. We refer to a matrix as having *full column rank* if its rank equals its number of columns. Analogously, a matrix has *full row rank* if its rank equals its number of rows.

Next, we define some concepts. A mode-j vector of an $I_1 \times I_2 \times \cdots \times I_n$ tensor is defined as an $I_j \times 1$ vector that is obtained by varying the jth index and keeping the other indices fixed. A mode-j matrix unfolding of a tensor is defined as a matrix containing all mode-j vectors as either rows or columns. For the decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ in (1.1), we define the mode-j matrix unfolding as

$$\left(\bigodot_{i \neq j}^{n} \mathbf{A}^{(i)}\right)(\mathbf{A}^{(j)})^T, \tag{2.1}$$

where $\bigodot$ denote a series of (columnwise) Khatri–Rao products.

For decompositions with some form of symmetry, we introduce the following notation to define the form of symmetry. Let $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_p\}$ with each $\mathcal{E}_q \subseteq \{1, \ldots, n\}$ containing mode numbers for which the component matrices are identical; i.e., for $i \neq j$ and some q,

$$i, j \in \mathcal{E}_q \quad \Leftrightarrow \quad \mathbf{A}^{(i)} = \mathbf{A}^{(j)}. \tag{2.2}$$

Hence, $j \in \mathcal{E}$ if and only if $\mathbf{A}^{(j)}$ is identical to some other component matrix. We require that $\mathcal{E}_q \cap \mathcal{E}_u = \varnothing$ for $q \neq u$.

3. Necessary uniqueness conditions. Here, we prove two necessary uniqueness conditions for decompositions with some form of symmetry. The conditions are symmetric analogues of necessary uniqueness conditions proven in [38]. The asymmetric conditions of [38] are stated in section 3.1, while section 3.2 contains the symmetric results.

3.1. Asymmetric decompositions. Below, we state two necessary uniqueness results by [38] for an asymmetric decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$.

LEMMA 3.1. *If* $\text{rank}(\bigodot_{i \neq j}^{n} \mathbf{A}^{(i)}) < R$ *for some* $j \in \{1, \ldots, n\}$, $n \geq 3$, *then the decomposition* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *is not unique up to permutation and scaling. Moreover, an alternative decomposition into* $R - 1$ *rank*-1 *terms exists.* ∎

LEMMA 3.2. *If the decomposition* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$, $n \geq 3$, *contains* $n - 2$ *distinct component matrices that have columns* s *and* t *proportional,* $s \neq t$, *then the decomposition is not unique up to permutation and scaling.* ∎

Lemma 3.1 states that any Khatri–Rao product of all but one component matrix has full column rank if the decomposition is unique. For $n = 3$, this result is due to Liu and Sidiropoulos [25]. For $n = 3$, Lemma 3.2 states the well-known necessary uniqueness condition $k_{\mathbf{A}^{(j)}} \geq 2$ for $j = 1, 2, 3$.

3.2. Decompositions with some form of symmetry. Our symmetric analogue of Lemma 3.1 is the following.

LEMMA 3.3. *Let the decomposition* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *have some form of symmetry,* $n \geq 3$. *If* $\text{rank}(\bigodot_{i \neq j}^{n} \mathbf{A}^{(i)}) < R$ *for some* $j \in \{1, \ldots, n\}$, *and* $j \notin \mathcal{E}$, *then the decomposition is not unique up to permutation and scaling. Moreover, an alternative decomposition into* $R - 1$ *rank*-1 *terms exists with the same form of symmetry.*

Proof. The proof is identical to the proof of Stegeman [38, Lemma 3.1]. We repeat it for completeness. Suppose $(\bigodot_{i \neq j}^{n} \mathbf{A}^{(i)})\mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$. Then the mode-j

matrix unfolding of the decomposition satisfies

$$\left(\bigodot_{i\neq j}^{n}\mathbf{A}^{(i)}\right)(\mathbf{A}^{(j)})^T=\left(\bigodot_{i\neq j}^{n}\mathbf{A}^{(i)}\right)(\mathbf{A}^{(j)}+\mathbf{y}\mathbf{x}^T)^T \tag{3.1}$$

for any vector $\mathbf{y}$. Hence, in the decomposition we may replace $\mathbf{A}^{(j)}$ by $(\mathbf{A}^{(j)}+\mathbf{y}\mathbf{x}^T)$ for any vector $\mathbf{y}$. This proves nonuniqueness. Moreover, we can choose $\mathbf{y}$ such that one column, say column p, of $(\mathbf{A}^{(j)}+\mathbf{y}\mathbf{x}^T)$ vanishes. Hence, a decomposition into $R-1$ rank-1 terms can be obtained by deleting columns $\mathbf{a}_p^{(i)}$ from each component matrix $\mathbf{A}^{(i)}$, $i\neq j$, and replacing $\mathbf{A}^{(j)}$ by $(\mathbf{A}^{(j)}+\mathbf{y}\mathbf{x}^T)$ with its all-zero column p deleted. ∎

In the proof of Lemma 3.3 an alternative decomposition is constructed by changing only $\mathbf{A}^{(j)}$. When $j\notin\mathcal{E}$, the alternative decomposition features the same form of symmetry as the original decomposition. This raises the question of whether Lemma 3.3 is still true for $j\in\mathcal{E}$. In that case, changing only $\mathbf{A}^{(j)}$ yields an alternative decomposition with a different form of symmetry (or none at all). We were not able to prove Lemma 3.3 for this case. However, we also have not found a counterexample; that is, a unique decomposition for which $(\bigodot_{i\neq j}^{n}\mathbf{A}^{(i)})$ has rank less than R and $j\in\mathcal{E}$. Hence, this issue remains an open question.

For $j=n=3$ and $\mathbf{A}^{(1)}=\mathbf{A}^{(2)}$, it is conjectured that the existence of an asymmetric nonequivalent alternative decomposition implies the existence of a symmetric nonequivalent alternative decomposition. If this is true, then $\operatorname{rank}(\mathbf{A}^{(1)}\odot\mathbf{A}^{(3)})<R$ implies nonuniqueness. No counterexample to this conjecture has been found so far. Proofs of the conjecture for various cases can be found in Ten Berge, Sidiropoulos, and Rocci [44] and Ten Berge, Stegeman, and Bennani Dosse [45].

Next, we show that Lemma 3.2 remains true in the symmetric case. Its proof, however, is more complicated.

LEMMA 3.4. *Let the decomposition* $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n)})$ *have some form of symmetry,* $n\geq 3$. *If there exist* $n-2$ *distinct component matrices that have columns* s *and* t *proportional,* $s\neq t$, *then the decomposition is not unique up to permutation and scaling.*

Proof. Without loss of generality, let $\mathbf{a}_s^{(j)}=\alpha^{(j)}\mathbf{a}_t^{(j)}$ for $j=1,\ldots,n-2$. If $n-1\notin\mathcal{E}$ and $n\notin\mathcal{E}$, then the proof is identical to Stegeman [38, Lemma 3.2]. We repeat this proof for completeness. For the rank-1 terms s and t of the decomposition we have

$$\begin{aligned}&\mathbf{a}_s^{(1)}\circ\cdots\circ\mathbf{a}_s^{(n)}+\mathbf{a}_t^{(1)}\circ\cdots\circ\mathbf{a}_t^{(n)}=\mathbf{a}_t^{(1)}\circ\cdots\circ\mathbf{a}_t^{(n-2)}\circ[\tilde{\alpha}\mathbf{a}_s^{(n-1)}|\mathbf{a}_t^{(n-1)}][\mathbf{a}_s^{(n)}|\mathbf{a}_t^{(n)}]^T\\&\quad=\mathbf{a}_t^{(1)}\circ\cdots\circ\mathbf{a}_t^{(n-2)}\circ[\tilde{\alpha}\mathbf{a}_s^{(n-1)}|\mathbf{a}_t^{(n-1)}]\mathbf{U}([\mathbf{a}_s^{(n)}|\mathbf{a}_t^{(n)}]\mathbf{U}^{-T})^T\end{aligned} \tag{3.2}$$

with $\tilde{\alpha}=\prod_{j=1}^{n-2}\alpha^{(j)}$ and $\mathbf{U}$ a nonsingular 2×2 matrix. Since $\mathbf{U}$ is not limited to the product of a permutation matrix and a nonsingular diagonal matrix, (3.2) implies nonuniqueness. As can be seen, the nonuniqueness of the matrix decomposition (second order) is used here. Also, since the first $n-2$ component matrices are changed identically (by replacing column s by column t), the alternative decomposition features the same form of symmetry as the original decomposition.

Next, suppose $n-1\in\mathcal{E}$ and $n\notin\mathcal{E}$. Then columns s and t of $\mathbf{A}^{(j)}$ are proportional, $j=1,\ldots,n-1$. This implies that columns s and t of $(\bigodot_{j=1}^{n-1}\mathbf{A}^{(j)})$ are proportional. Hence, the latter has rank less than R. By Lemma 3.3, the decomposition is not unique, and an alternative decomposition (with the same form of symmetry) exists with $R-1$ components.

Next, suppose $n-1\in\mathcal{E}$ and $n\in\mathcal{E}$ with $\mathbf{A}^{(n-1)}\neq\mathbf{A}^{(n)}$. Then columns s and t of $\mathbf{A}^{(j)}$ are proportional, $j=1,\ldots,n$. Let $\mathbf{a}_s^{(j)}=\alpha^{(j)}\mathbf{a}_t^{(j)}$ also for $j=n-1,n$. This implies that

$$\mathbf{a}_s^{(1)} \circ \cdots \circ \mathbf{a}_s^{(n)} + \mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n)} = (1+\bar{\alpha})(\mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n)}) \tag{3.3}$$

with $\bar{\alpha} = \prod_{j=1}^{n} \alpha^{(j)}$. Hence, the sum of these two rank-1 terms is a rank-1 term itself. An alternative decomposition featuring $R-1$ components and the same form of symmetry can be constructed as follows. In each component matrix we delete column t. If $\mathcal{E} \neq \{1, \ldots, n\}$, then there exists $i \notin \mathcal{E}$. We replace $\mathbf{a}_s^{(i)}$ by $(1+\bar{\alpha})\mathbf{a}_t^{(i)}$, and we replace $\mathbf{a}_s^{(j)}$ by $\mathbf{a}_t^{(j)}$ for $j \neq i$. Next, suppose $\mathcal{E} = \{1, \ldots, n\}$. If there exists $\mathcal{E}_q$ with odd cardinality k, then we replace $\mathbf{a}_s^{(i)}$ by $(1+\bar{\alpha})^{(1/k)}\mathbf{a}_t^{(i)}$ for $i \in \mathcal{E}_q$, and we replace $\mathbf{a}_s^{(j)}$ by $\mathbf{a}_t^{(j)}$ for $j \notin \mathcal{E}_q$. If there does not exist $\mathcal{E}_q$ with odd cardinality, then $\bar{\alpha}$ is a product of squared numbers and, hence, $\bar{\alpha} \geq 0$. In this case, we replace $\mathbf{a}_s^{(j)}$ by $(1+\bar{\alpha})^{(1/n)}\mathbf{a}_t^{(j)}$ for $j = 1, \ldots, n$.

It remains to consider the case $n-1 \in \mathcal{E}$ and $n \in \mathcal{E}$ with $\mathbf{A}^{(n-1)} = \mathbf{A}^{(n)}$. Let $n-1$, $n \in \mathcal{E}_q$. If $j \in \mathcal{E}_q$ for some $1 \leq j \leq n-2$, then (3.3) holds, and we can proceed as above. The same is true when $\mathbf{a}_s^{(n)}$ and $\mathbf{a}_t^{(n)}$ are proportional. Next, suppose $\mathcal{E}_q = \{n-1, n\}$ and $\mathbf{a}_s^{(n)}$ and $\mathbf{a}_t^{(n)}$ are not proportional (or all-zero). Let

$$\mathbf{S} = \begin{bmatrix} \sqrt{1-\tilde{\alpha}\beta^2} & -\tilde{\alpha}\beta \\ \beta & \sqrt{1-\tilde{\alpha}\beta^2} \end{bmatrix} \tag{3.4}$$

with $\tilde{\alpha} = \prod_{j=1}^{n-2} \alpha^{(j)}$ as above and β chosen such that $1-\tilde{\alpha}\beta^2 > 0$. Let $[\bar{\mathbf{a}}_s^{(n)}|\bar{\mathbf{a}}_t^{(n)}] = [\mathbf{a}_s^{(n)}|\mathbf{a}_t^{(n)}]\mathbf{S}$. It can be verified that $\mathbf{S}\mathrm{diag}(\tilde{\alpha}, 1)\mathbf{S}^T = \mathrm{diag}(\tilde{\alpha}, 1)$. This implies that, analogous to (3.2), we have

$$\begin{aligned} \mathbf{a}_s^{(1)} \circ \cdots \circ \mathbf{a}_s^{(n)} + \mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n)} &= \mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n-2)} \circ [\tilde{\alpha}\mathbf{a}_s^{(n)}|\mathbf{a}_t^{(n)}][\mathbf{a}_s^{(n)}|\mathbf{a}_t^{(n)}]^T \\ &= \mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n-2)} \circ [\tilde{\alpha}\bar{\mathbf{a}}_s^{(n-1)}|\bar{\mathbf{a}}_t^{(n-1)}][\bar{\mathbf{a}}_s^{(n)}|\bar{\mathbf{a}}_t^{(n)}]^T, \\ &= \mathbf{a}_s^{(1)} \circ \cdots \circ \mathbf{a}_s^{(n-2)} \circ \bar{\mathbf{a}}_s^{(n)} \circ \bar{\mathbf{a}}_s^{(n)} + \bar{\mathbf{a}}_t^{(1)} \circ \cdots \circ \bar{\mathbf{a}}_t^{(n-2)} \circ \bar{\mathbf{a}}_t^{(n)} \circ \bar{\mathbf{a}}_t^{(n)}, \end{aligned} \tag{3.5}$$

which shows nonuniqueness of the decomposition. Note that the alternative decomposition is constructed by changing only $\mathbf{A}^{(n-1)} = \mathbf{A}^{(n)}$ and features the same form of symmetry as the original decomposition. This completes the proof. ∎

4. Uniqueness conditions for the case rank$(\mathbf{A}^{(n)}) = R$. Here, we consider uniqueness conditions for a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ with some component matrix $\mathbf{A}^{(j)}$ having full column rank R. For convenience, we set $j = n$. Section 4.1 states the uniqueness conditions of [38] for the asymmetric case. Section 4.2 contains the analogues for decompositions with some form of symmetry. Also, we show that all alternative decompositions have the same form of symmetry as the original decomposition when $\mathrm{rank}(\mathbf{A}^{(n)}) = R$ and $\mathrm{rank}(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)}) = R$.

4.1. Asymmetric decompositions. We assume that $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ has full column rank R, and we denote an alternative decomposition as $(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)})$. Equating the mode-$n$ matrix unfoldings of the two decompositions yields

$$(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})(\mathbf{A}^{(n)})^T = (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})(\mathbf{B}^{(n)})^T. \tag{4.1}$$

Since the left-hand side of (4.1) has rank R, it also follows that the two matrices on the right-hand side of (4.1) have rank R. Uniqueness of the decomposition is not affected by premultiplying a component matrix by a nonsingular matrix. Let $\mathbf{S}$ be nonsingular such that $\mathbf{S}\mathbf{A}^{(n)} = \begin{bmatrix} \mathbf{I}_R \\ \mathbf{O} \end{bmatrix}$. When replacing $\mathbf{A}^{(n)}$ in (4.1) by $\mathbf{S}\mathbf{A}^{(n)}$, the full column rank of

$(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})$ guarantees that the last $I_n - R$ rows of $\mathbf{B}^{(n)}$ are all-zero. Hence, without loss of generality we may set $\mathbf{A}^{(n)} = \mathbf{I}_R$ (and $I_n = R$) and $\mathbf{B}^{(n)}$ square and nonsingular. For $n = 3$ this was shown by Ten Berge and Sidiropoulos [43]. We rewrite (4.1) as

$$(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})(\mathbf{B}^{(n)})^{-T} = (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)}). \tag{4.2}$$

The following result of [38] shows that uniqueness holds if and only if each linear combination of the columns of $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ has at most one nonzero coefficient. For $n = 3$ this is due to Jiang and Sidiropoulos [18]. Let $\omega(\cdot)$ denote the number of nonzero entries of a vector.

THEOREM 4.1. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$, $n \geq 3$, *be a decomposition with* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$. *Then the decomposition is unique up to permutation and scaling if and only if, for any vector* $\mathbf{d} \in \mathbb{R}^R$,

$$(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\mathbf{d} = (\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{n-1}) \quad \text{implies} \quad \omega(\mathbf{d}) \leq 1. \qquad \square \tag{4.3}$$

Condition (4.3) implies full column rank of $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$. Condition (4.3) is not easy to check. Reshaping it into $I_1 \times \cdots \times I_{n-1}$ tensor form yields

$$\operatorname{rank}\left(\underline{\mathbf{Y}} = \sum_{r=1}^{R} d_r(\mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(n-1)})\right) \leq 1 \quad \text{implies} \quad \omega(\mathbf{d}) \leq 1 \tag{4.4}$$

with $\mathbf{d} = (d_1, d_2, \ldots, d_R)^T$. Stegeman [38, Lemma 4.6] shows that an $(n-1)$th order tensor has rank at most 1 if and only if its mode-j matrix unfolding has rank at most 1, $j = 1, \ldots, n-1$. Hence, all distinct 2×2 minors of the mode-j matrix unfolding of $\underline{\mathbf{Y}}$ should be equal to zero for $j = 1, \ldots, n-1$. This can be written as

$$\mathbf{U}_j^{(n-1)}\tilde{\mathbf{d}} = m\left(\bigodot_{i \neq j}^{n-1} \mathbf{A}^{(i)}\right) \odot m(\mathbf{A}^{(j)})\tilde{\mathbf{d}} = \mathbf{0}, \qquad j = 1, \ldots, n-1, \tag{4.5}$$

where $\tilde{\mathbf{d}} = (d_1 d_2, d_1 d_3, \ldots, d_{R-1} d_R)^T$, and $m(\cdot)$ is defined as follows.

DEFINITION 4.2. *For an* $I \times R$ *matrix* $\mathbf{A}$, *let the* $I(I-1)/2 \times R(R-1)/2$ *matrix* $m(\mathbf{A})$ *have entries*

$$\det\begin{pmatrix} a_{ig} & a_{ih} \\ a_{jg} & a_{jh} \end{pmatrix} \quad \text{with} \quad 1 \leq i < j \leq I \quad \text{and} \quad 1 \leq g < h \leq R, \tag{4.6}$$

where in each row of $m(\mathbf{A})$ *the value of* (i, j) *is fixed and in each column of* $m(\mathbf{A})$ *the value of* (g, h) *is fixed. The columns of* $m(\mathbf{A})$ *are ordered such that index* g *runs slower than* h. *The rows of* $m(\mathbf{A})$ *are ordered such that index* i *runs slower than* j. $\square$

By defining

$$\mathbf{U}^{(n-1)} = \begin{bmatrix} \mathbf{U}_1^{(n-1)} \\ \vdots \\ \mathbf{U}_{n-1}^{(n-1)} \end{bmatrix}, \tag{4.7}$$

the following equivalent of Theorem 4.1 is obtained. For $n = 3$, this result is due to Jiang and Sidiropoulos [18].

THEOREM 4.3. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$, $n \geq 3$, *be a decomposition with* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$. *Then the decomposition is unique up to permutation and scaling if and only if, for any vector* $\mathbf{d} \in \mathbb{R}^R$,

$$\mathbf{U}^{(n-1)}\tilde{\mathbf{d}} = \mathbf{0} \quad \text{implies} \quad \omega(\mathbf{d}) \leq 1, \tag{4.8}$$

where $\tilde{\mathbf{d}} = (d_1 d_2, d_1 d_3, \ldots, d_{R-1} d_R)^T$. ∎

From the form of $\tilde{\mathbf{d}}$ it can be seen that $\tilde{\mathbf{d}} = \mathbf{0}$ implies $\omega(\mathbf{d}) \leq 1$. Hence, $\mathbf{U}^{(n-1)}$ having full column rank is sufficient for condition (4.8) to hold.

COROLLARY 4.4. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$, $n \geq 3$, *be a decomposition with* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$. *Then the decomposition is unique up to permutation and scaling if* $\mathbf{U}^{(n-1)}$ *has full column rank.* ∎

Contrary to (4.3), the condition of $\mathbf{U}^{(n-1)}$ having full column rank is easy to check. Corollary 4.4 was proven independently for $n = 3$ and $n = 4$ by De Lathauwer [7].

4.2. Decompositions with some form of symmetry. Here, we consider uniqueness conditions for a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ with some form of symmetry and $\operatorname{rank}(\mathbf{A}^{(n)}) = R$. If $n \in \mathcal{E}$, then at least one other component matrix has rank R as well. Condition (1.3) implies uniqueness when the sum of the other $n - 2$ k-ranks equals at least $n - 1$ (and no k-rank equals zero). This condition is very mild. In the following we assume $n \notin \mathcal{E}$. For $n \in \mathcal{E}$, we refer to condition (1.3).

Before we prove uniqueness conditions, we consider the type of alternative decompositions that are possible. Theorem 4.5 below states that, up to scaling indeterminacies, any alternative decomposition will feature the same form of symmetry when $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ has full column rank R. For $n = 3$ and $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$, Theorem 4.5 complements the results of Ten Berge, Sidiropoulose, and Rocci [44] and Ten Berge, Stegeman, and Dosse [45]. When $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ has rank less than R, nonuniqueness follows from Lemma 3.3.

THEOREM 4.5. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *be a decomposition with some form of symmetry,* $n \geq 3$. *Let* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$ *and* $n \notin \mathcal{E}$. *If* $\operatorname{rank}(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)}) = R$, *then any alternative decomposition* $(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)})$ *satisfies* $\mathbf{B}^{(i)} = \mathbf{B}^{(j)}\mathbf{\Lambda}^{(ij)}$ *if* $\mathbf{A}^{(i)} = \mathbf{A}^{(j)}$ *for* $1 \leq i, j \leq n - 1$, *with* $\mathbf{\Lambda}^{(ij)}$ *being a nonsingular diagonal matrix.*

Proof. As in section 4.1, we assume without loss of generality that $\mathbf{A}^{(n)} = \mathbf{I}_R$ (and $I_n = R$). We focus on (4.2) in which $\mathbf{B}^{(n)}$ is nonsingular and $(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})$ has full column rank. Let $\mathbf{d}$ be an arbitrary column of $(\mathbf{B}^{(n)})^{-T}$, and let $(\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{n-1})$ be an arbitrary column of $(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})$. We reshape the equation $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\mathbf{d} = (\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{n-1})$ in $(n-1)$th order tensor form as

$$\underline{\mathbf{Y}} = \sum_{r=1}^{R} d_r(\mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(n-1)}) = (\mathbf{f}_1 \circ \cdots \circ \mathbf{f}_{n-1}). \tag{4.9}$$

Full column rank of $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ guarantees that $\underline{\mathbf{Y}}$ is not all-zero unless $\mathbf{d} = \mathbf{0}$. The latter is impossible since $\mathbf{d}$ is a column of the nonsingular $(\mathbf{B}^{(n)})^{-T}$. Since $\underline{\mathbf{Y}}$ is not all-zero, none of the $\mathbf{f}_j$ are all-zero. Moreover, the rank-1 tensor on the right-hand side of (4.9) should have the same form of symmetry as $\underline{\mathbf{Y}}$.

Let $\mathcal{E}_q = \{j_1, \ldots, j_k\}$; i.e., $\mathbf{A}^{(j_1)} = \cdots = \mathbf{A}^{(j_k)}$. Then (4.9) implies that the kth order rank-1 tensor $(\mathbf{f}_{j_1} \circ \cdots \circ \mathbf{f}_{j_k})$ is symmetric in all k modes. Since all $\mathbf{f}_j \neq \mathbf{0}$, it follows that

$\mathbf{f}_{j_1}, \ldots, \mathbf{f}_{j_k}$ are proportional. The latter are column c of $\mathbf{B}^{(j_1)}, \ldots, \mathbf{B}^{(j_k)}$ with c arbitrary. This completes the proof. ∎

The alternative decompositions in Theorem 4.5 can be rescaled to feature the same form of symmetry as the original decomposition (i.e., $\mathbf{B}^{(i)} = \mathbf{B}^{(j)}$ if $\mathbf{A}^{(i)} = \mathbf{A}^{(j)}$). This can be done by incorporating all the $\mathbf{\Lambda}^{(ij)}$ (containing the constants of proportionality) in matrix $\mathbf{B}^{(n)}$.

In Theorem 4.6 below, we prove the symmetric analogue of the necessary and sufficient uniqueness condition in Theorem 4.1. Considering the result of Theorem 4.5, it may not be surprising that, without loss of generality, we may take $\mathbf{f}_i = \mathbf{f}_j$ if $\mathbf{A}^{(i)} = \mathbf{A}^{(j)}$ in condition (4.3).

THEOREM 4.6. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *be a decomposition with some form of symmetry,* $n \geq 3$. *Let* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$ *and* $n \notin \mathcal{E}$. *Then the decomposition is unique up to permutation and scaling if and only if, for any vector* $\mathbf{d} \in \mathbb{R}^R$,

$$(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\mathbf{d} = (\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{n-1}) \quad \text{implies} \quad \omega(\mathbf{d}) \leq 1, \tag{4.10}$$

where $\mathbf{f}_i = \mathbf{f}_j$ *if* $\mathbf{A}^{(i)} = \mathbf{A}^{(j)}$, $1 \leq i, j \leq n-1$.

Proof. First, we prove that (4.10) is a sufficient condition for uniqueness. Condition (4.10) implies full column rank of $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$. Indeed, rank deficiency implies that either $\omega(\mathbf{d}) \geq 2$ and $\mathbf{f}_1 = \mathbf{0}$ is possible or that $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ has an all-zero column, which makes $\omega(\mathbf{d}) = 2$ possible. As in the proof of Theorem 4.5, we focus on (4.2). Condition (4.10) implies that each column of $(\mathbf{B}^{(n)})^{-T}$ contains at most one nonzero entry. Since $\mathbf{B}^{(n)}$ is nonsingular, each column has exactly one nonzero entry. Moreover, $(\mathbf{B}^{(n)})^{-T} = \mathbf{\Pi}\mathbf{\Lambda}_n^{-1}$, where $\mathbf{\Pi}$ is a permutation matrix, and $\mathbf{\Lambda}_n$ is a nonsingular diagonal matrix. Hence, we have $\mathbf{B}^{(n)} = \mathbf{\Pi}\mathbf{\Lambda}_n$, and it follows that

$$(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)}) = (\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\mathbf{\Pi}\mathbf{\Lambda}_n^{-1} = (\mathbf{A}^{(1)}\mathbf{\Pi} \odot \cdots \odot \mathbf{A}^{(n-1)}\mathbf{\Pi})\mathbf{\Lambda}_n^{-1}. \tag{4.11}$$

Hence, each column of $(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})$ is a rescaled column of $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$. Each such column r can be interpreted as a vectorized $(n-1)$th order tensor that is the outer product of $\mathbf{b}_r^{(1)}, \ldots, \mathbf{b}_r^{(n-1)}$. Rewriting one column in this form yields

$$\mathbf{b}_r^{(1)} \circ \cdots \circ \mathbf{b}_r^{(n-1)} = \lambda(\mathbf{a}_q^{(1)} \circ \cdots \circ \mathbf{a}_q^{(n-1)}), \tag{4.12}$$

where, for fixed r, the value of q is given by the permutation $\mathbf{\Pi}$, and $\lambda \neq 0$ is the corresponding diagonal entry of $\mathbf{\Lambda}_n^{-1}$. Since none of the columns in (4.12) is all-zero, it follows that $\mathbf{b}_r^{(j)}$ is proportional to $\mathbf{a}_q^{(j)}$ for $j = 1, \ldots, n-1$. This implies $\mathbf{B}^{(j)} = \mathbf{A}^{(j)}\mathbf{\Pi}\mathbf{\Lambda}_j$, $j = 1, \ldots, n-1$, for nonsingular diagonal matrices $\mathbf{\Lambda}_j$. Since $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ has full column rank, (4.11) implies that $\prod_{j=1}^{n-1} \mathbf{\Lambda}_j = \mathbf{\Lambda}_n^{-1}$. Hence, the decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ is unique up to permutation and scaling. This shows the sufficiency of condition (4.10). Note that the signs of the diagonal entries of $\mathbf{\Lambda}_n$ are required to accommodate the form of symmetry in the decomposition. For example, if $n = 3$ and $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$, then (4.12) implies $\lambda > 0$.

Next, we prove necessity of (4.10) analogous to the proof of Stegeman [38, Theorem 4.2]. As in section 4.1, we set $\mathbf{A}^{(n)} = \mathbf{I}_R$ without loss of generality. Suppose $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\mathbf{d} = (\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{n-1})$ for some vector $\mathbf{d}$ with $\omega(\mathbf{d}) \geq 2$ and $\mathbf{f}_i = \mathbf{f}_j$ if $\mathbf{A}^{(i)} = \mathbf{A}^{(j)}$. Let $d_p \neq 0$, and set $\mathbf{B}^{(j)}$ equal to $\mathbf{A}^{(j)}$ with column p replaced by $\mathbf{f}_j$, $j = 1, \ldots, n-1$. Then $(\mathbf{a}_p^{(1)} \otimes \cdots \otimes \mathbf{a}_p^{(n-1)}) = (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})\mathbf{g}$ for some vector $\mathbf{g}$ with $\omega(\mathbf{g}) \geq 2$. Let $\mathbf{B}^{(n)}$ be equal to $\mathbf{I}_R$ with row p replaced by $\mathbf{g}^T$. We have

$$(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)}) = (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})(\mathbf{B}^{(n)})^T. \tag{4.13}$$

Moreover, if $\mathbf{A}^{(i)} = \mathbf{A}^{(j)}$, then $\mathbf{B}^{(i)} = \mathbf{B}^{(j)}$ for $1 \leq i, j \leq n-1$. Together with $n \notin \mathcal{E}$, this shows that $(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)})$ is an alternative decomposition featuring the same form of symmetry as $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)}, \mathbf{I}_R)$. Since the $\mathbf{B}^{(j)}$ are not rescaled column permutations of the $\mathbf{A}^{(j)}$, the nonuniqueness of the decomposition follows. ∎

When comparing conditions (4.3) and (4.10) for a decomposition featuring some form of symmetry, the following can be observed. By the logical forms of the conditions, it is clear that (4.3) implies (4.10). If (4.3) does not hold, then a nonequivalent alternative decomposition exists. By Theorem 4.5, this alternative decomposition features the same form of symmetry as the original decomposition. Hence, we have nonuniqueness which also implies that (4.10) does not hold. This yields the following corollary.

COROLLARY 4.7. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *be a decomposition with some form of symmetry,* $n \geq 3$. *Let* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$ *and* $n \notin \mathcal{E}$. *Then condition* (4.3) *holds if and only if condition* (4.10) *holds.* ∎

Since $\mathbf{U}^{(n-1)}$ having full column rank implies condition (4.8), which is equivalent to condition (4.3) by Theorem 4.3, we obtain the following analogue of Corollary 4.4.

COROLLARY 4.8. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *be a decomposition with some form of symmetry,* $n \geq 3$. *Let* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$ *and* $n \notin \mathcal{E}$. *Then the decomposition is unique up to permutation and scaling if the matrix* $\mathbf{U}^{(n-1)}$ *has full column rank.* ∎

5. Generic uniqueness conditions for the case rank$(\mathbf{A}^{(n)}) = R$. Here, we consider the matrix $\mathbf{U}^{(n-1)}$ when $\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)}$ are generic. In [38] bounds on R are obtained in terms of $I_1, \ldots, I_{n-1}$ such that $\mathbf{U}^{(n-1)}$ has full column rank under these bounds. In view of Corollary 4.4, these are called generic uniqueness conditions. In section 5.1, we give an overview of these conditions as proven in [38] for the asymmetric case. In section 5.2, we present generic uniqueness bounds for $n = 3, 4, 5$ and several forms of symmetry. Also, we compare the bounds under symmetry to the bounds for the asymmetric case.

5.1. Asymmetric decompositions. Each row of $\mathbf{U}^{(n-1)}$ in (4.7) corresponds to a 2×2 minor of a matrix unfolding of the $(n-1)$th order tensor $\underline{\mathbf{Y}}$ with decomposition

$$\underline{\mathbf{Y}} = \sum_{r=1}^{R} d_r(\mathbf{a}_r^{(1)} \circ \cdots \circ \mathbf{a}_r^{(n-1)}). \tag{5.1}$$

According to condition (4.4), the tensor $\underline{\mathbf{Y}}$ needs to have rank at most 1. This is guaranteed when all 2×2 minors of all its matrix unfoldings are zero. However, checking all distinct 2×2 minors of all matrix unfoldings of $\underline{\mathbf{Y}}$ is not needed. Some 2×2 minors are redundant regardless of the entries of $\underline{\mathbf{Y}}$ or the sizes of $I_1, \ldots, I_{n-1}$ and R. Since each 2×2 minor corresponds to a row in $\mathbf{U}^{(n-1)}$, a redundant minor corresponds to a redundant row of $\mathbf{U}^{(n-1)}$. When these redundant rows are deleted from $\mathbf{U}^{(n-1)}$, then it has full column rank if it is square or vertical, and $\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)}$ are generic. For $n = 3$ and $n = 4$ this was shown by De Lathauwer [7]. The following generalization to arbitrary $n \geq 3$ is due to Stegeman [38]. Let the numbers $Q_{(m,n)}$ be given by

$$Q_{(m,n)} = \sum_{\mathcal{S}_m} \prod_{j \in \mathcal{S}_m} \frac{I_j(I_j - 1)}{2} \prod_{j \notin \mathcal{S}_m} I_j, \tag{5.2}$$

where the summation is over all subsets $\mathcal{S}_m$ of $\{1, \ldots, n-1\}$ containing m distinct elements. If $m = n - 1$, then we set $\prod_{j \notin \mathcal{S}_m} I_j = 1$.

THEOREM 5.1. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$, $n \geq 3$, *be a decomposition with generic* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$ *and* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$. *Then* $\mathbf{U}^{(n-1)}$ *has full column rank if*

$$\frac{R(R-1)}{2} \leq \sum_{m=2}^{n-1} (2^{m-1} - 1) Q_{(m,n)}. \tag{5.3}$$

Hence, the decomposition is unique up to permutation and scaling if (5.3) *holds.* ∎

The matrix $\mathbf{U}^{(n-1)}$ has $R(R-1)/2$ columns. The right-hand side of (5.3) represents its number of nonredundant rows.

5.2. Decompositions with some form of symmetry. In this section, we present generic uniqueness bounds for $n = 3, 4, 5$ and several forms of symmetry. The symmetry in the decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$ is of the same form as the symmetry in $\underline{\mathbf{Y}}$ in (5.1). As in [38], the number of nonredundant rows of $\mathbf{U}^{(n-1)}$ is determined as follows. Each row of $\mathbf{U}^{(n-1)}$ corresponds to a 2×2 minor of a matrix unfolding of $\underline{\mathbf{Y}}$. A 2×2 minor of a matrix unfolding of $\underline{\mathbf{Y}}$ corresponds to an equation $y_{\mathcal{A}} y_{\mathcal{D}} = y_{\mathcal{B}} y_{\mathcal{C}}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ contain $n-1$ indices. Since we do not know which entries of $\underline{\mathbf{Y}}$ are nonzero, redundant minors can only be identified by determining identical terms in the equations of the minors. Here, a term is the product of two entries of $\underline{\mathbf{Y}}$, as in $y_{\mathcal{A}} y_{\mathcal{D}}$.

In this way, we are able to determine the number of nonredundant rows of $\mathbf{U}^{(n-1)}$. When the latter equals K, the generic uniqueness bound is of the form $R(R-1)/2 \leq K$, since $\mathbf{U}^{(n-1)}$ has $R(R-1)/2$ columns. Let

$$\Phi(I) = \frac{I(I-1)}{4}\left(\frac{I(I-1)}{2} + 1\right) - \binom{I}{4}, \tag{5.4}$$

$$\Psi(I) = 2\Phi(I) + \binom{I}{4}, \qquad \Omega(I) = 4\Phi(I) + 3\binom{I}{4}, \tag{5.5}$$

$$\Delta(I, J) = 2\left(\Phi(I) + \binom{I}{4}\right)\left(\Phi(J) + \binom{J}{4}\right) - \binom{I}{4}\binom{J}{4}, \tag{5.6}$$

where the terms $\binom{x}{4}$ appear only if $x \geq 4$. Our results are the following analogues of Theorem 5.1. Theorem 5.2 below concerns the case $n = 3$ and $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$, and it was conjectured and partly proven in Stegeman, Ten Berge, and De Lathauwer [40]. As discussed in section 1, there are a lot of applications using this decomposition. Theorems 5.3 and 5.4 are generalizations to $n = 4$ and $n = 5$, respectively. Theorem 5.5 is a special case of Theorem 5.4 in which there is symmetry in modes 1 and 2, but also in modes 3 and 4. This concerns the third order decomposition combined with ICA (resulting in a fifth order decomposition) of De Vos et al. [12]. The proofs of the theorems below can be found in the appendices.

THEOREM 5.2. *Let* $(\mathbf{A}^{(1)}, \mathbf{A}^{(1)}, \mathbf{A}^{(3)})$ *be a decomposition with generic* $\mathbf{A}^{(1)}$ *and* $\operatorname{rank}(\mathbf{A}^{(3)}) = R$. *Then* $\mathbf{U}^{(2)}$ *has full column rank if*

$$\frac{R(R-1)}{2} \leq \Phi(I_1). \tag{5.7}$$

Hence, the decomposition is unique up to permutation and scaling if (5.7) *holds.* ∎

THEOREM 5.3. *Let* $(\mathbf{A}^{(1)}, \mathbf{A}^{(1)}, \mathbf{A}^{(3)}, \mathbf{A}^{(4)})$ *be a decomposition with generic* $(\mathbf{A}^{(1)}, \mathbf{A}^{(3)})$ *and* $\operatorname{rank}(\mathbf{A}^{(4)}) = R$. *Then* $\mathbf{U}^{(3)}$ *has full column rank if*

$$\frac{R(R-1)}{2} \leq I_3 \Phi(I_1) + \frac{I_3(I_3-1)}{2}\left(\frac{I_1^2(I_1-1)}{2} + \Psi(I_1)\right). \tag{5.8}$$

Hence, the decomposition is unique up to permutation and scaling if (5.8) *holds.* ∎

THEOREM 5.4. *Let* $(\mathbf{A}^{(1)}, \mathbf{A}^{(1)}, \mathbf{A}^{(3)}, \mathbf{A}^{(4)}, \mathbf{A}^{(5)})$ *be a decomposition with generic* $(\mathbf{A}^{(1)}, \mathbf{A}^{(3)}, \mathbf{A}^{(4)})$ *and* $\operatorname{rank}(\mathbf{A}^{(5)}) = R$. *Then* $\mathbf{U}^{(4)}$ *has full column rank if*

$$\begin{aligned}\frac{R(R-1)}{2} \leq{}& I_3 I_4 \Phi(I_1) + \left(I_3 \frac{I_4(I_4-1)}{2} + I_4 \frac{I_3(I_3-1)}{2}\right)\left(\frac{I_1^2(I_1-1)}{2} + \Psi(I_1)\right) \\ &+ \frac{I_3(I_3-1)}{2}\frac{I_4(I_4-1)}{2}\left(\frac{I_1(I_1+1)}{2} + \frac{3I_1^2(I_1-1)}{2} + \Omega(I_1)\right).\end{aligned} \tag{5.9}$$

Hence, the decomposition is unique up to permutation and scaling if (5.9) *holds.* ∎

THEOREM 5.5. *Let* $(\mathbf{A}^{(1)}, \mathbf{A}^{(1)}, \mathbf{A}^{(3)}, \mathbf{A}^{(3)}, \mathbf{A}^{(5)})$ *be a decomposition with generic* $(\mathbf{A}^{(1)}, \mathbf{A}^{(3)})$ *and* $\operatorname{rank}(\mathbf{A}^{(5)}) = R$. *Then* $\mathbf{U}^{(4)}$ *has full column rank if*

$$\begin{aligned}\frac{R(R-1)}{2} \leq{}& \frac{I_1(I_1+1)}{2}\Phi(I_3) + \frac{I_3(I_3+1)}{2}\Phi(I_1) + \frac{I_1^2(I_1-1)}{2}\frac{I_3^2(I_3-1)}{2} \\ &+ \frac{I_1^2(I_1-1)}{2}\Psi(I_3) + \frac{I_3^2(I_3-1)}{2}\Psi(I_1) + \Delta(I_1, I_3).\end{aligned} \tag{5.10}$$

Hence, the decomposition is unique up to permutation and scaling if (5.10) *holds.* ∎

Below, we compare the generic uniqueness bounds (5.7)–(5.10) to their asymmetric counterpart (5.3). For several cases, we compute the largest R that satisfies the bound. Also, we compute the largest R satisfying the generalization (1.3) of Kruskal's uniqueness condition (with $k_{\mathbf{A}^{(n)}} = R$ and $k_{\mathbf{A}^{(j)}} = \min(I_j, R) = I_j$, $j \leq n-1$). The results can be found in Table 1. As also observed in Stegeman [38], the bound (5.3) is a large improvement with respect to the bound obtained from (1.3). The most striking observation is done when comparing (5.3) to (5.7)–(5.10): the bounds on R are much lower in the presence of symmetry.

This is in line with the fact that the generic or typical rank of a tensor is lower when it has a form of symmetry; see Ten Berge, Sidiropoulos, and Rocci [44] and Comon et al. [5]. Uniqueness occurs for values of R lower than the generic or typical rank. For the cases in Table 1, typical rank results are known only for $n = 3$. Using the algorithm of [5], it can be verified that for asymmetric $4 \times 4 \times I_3$ tensors, the typical rank values increase from 8 to 16 when I_3 increases from 6 to 16 (for $I_3 > 16$ the typical rank is 16). The asymmetric generic uniqueness bound (5.3) is $R \leq 9$. For $4 \times 4 \times I_3$ tensors with symmetry in the first two modes, the typical rank values increase from 7 to 10

TABLE 1

Comparison of uniqueness bounds on R for generic decompositions $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *with* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$, *both without (columns* 3 *and* 4*) and with some form of symmetry (column* 5*).*

n	Size tensor	Bound on R from (1.3)	Bound on R from (5.3)	Bound on R with symmetry
$n=3$	$4 \times 4 \times I_3$, $I_3 \geq R$	$R \leq 6$	$R \leq 9$	$R \leq 6$ $(\mathbf{A}^{(1)} = \mathbf{A}^{(2)})$
$n=4$	$4 \times 4 \times 4 \times I_4$, $I_4 \geq R$	$R \leq 9$	$R \leq 46$	$R \leq 31$ $(\mathbf{A}^{(1)} = \mathbf{A}^{(2)})$
$n=5$	$4 \times 4 \times 4 \times 4 \times I_5$, $I_5 \geq R$	$R \leq 12$	$R \leq 214$	$R \leq 137$ $(\mathbf{A}^{(1)} = \mathbf{A}^{(2)})$
$n=5$	$4 \times 4 \times 4 \times 4 \times I_5$, $I_5 \geq R$	$R \leq 12$	$R \leq 214$	$R \leq 87$ $(\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)} = \mathbf{A}^{(4)})$

when I_3 increases from 6 to 10 (for $I_3 > 10$ the typical rank is 10), while the generic uniqueness bound (5.7) is $R \leq 6$.

6. Examples for $n = 3$ and $\mathbf{A^{(1)} = A^{(2)}}$. Here, we illustrate the results in sections 4 and 5 for the case $n = 3$ by means of several examples. We denote $\mathbf{A} = \mathbf{A}^{(1)} = \mathbf{A}^{(2)}$ and set $\mathbf{A}^{(3)} = \mathbf{I}_R$ without loss of generality.

Example 6.1. Let $R = 3$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad (\mathbf{A} \odot \mathbf{A}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}. \tag{6.1}$$

Then $(\mathbf{A} \odot \mathbf{A})\mathbf{d} = \mathbf{f} \otimes \mathbf{f}$ with $\mathbf{f} = (2, 2)^T$ and $\mathbf{d} = (2, -4, 2)^T$. Hence, condition (4.10) does not hold, and the decomposition $(\mathbf{A}, \mathbf{A}, \mathbf{I}_3)$ is not unique. Since $(\mathbf{A} \odot \mathbf{A})$ has full column rank, Theorem 4.5 implies that any alternative decomposition $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$ satisfies $\bar{\mathbf{B}} = \bar{\mathbf{A}}\mathbf{\Lambda}$ for some nonsingular diagonal matrix $\mathbf{\Lambda}$. □

Example 6.2. Let $R = 4$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{6.2}$$

It can be verified that $(\mathbf{A} \odot \mathbf{A})\mathbf{d} = \mathbf{f} \otimes \mathbf{f}$ yields the equations

$$d_j + d_4 = f_j^2, \quad j = 1, 2, 3, \quad d_4 = f_1 f_2 = f_1 f_3 = f_2 f_3. \tag{6.3}$$

When $\mathbf{f}$ contains no zeros, it follows that $f_1 = f_2 = f_3$ and $d_1 = d_2 = d_3 = 0$. When $f_1 = 0$, we get $d_4 = f_2 f_3 = 0$ and $d_1 = 0$. Hence, either $f_2 = 0$ (implying $d_2 = 0$) or $f_3 = 0$ (implying $d_3 = 0$), and $\omega(\mathbf{d}) \leq 1$ follows. When starting with $f_2 = 0$ or $f_3 = 0$, the same result is obtained. This shows that condition (4.10) holds, which implies uniqueness of $(\mathbf{A}, \mathbf{A}, \mathbf{I}_4)$. When the matrix $\mathbf{U}^{(2)}$ is computed, it can be verified that it has rank $R(R-1)/2 = 6$. Note that the right-hand side of (5.7) equals $\Phi(3) = 6$. Hence, after deleting redundant rows, $\mathbf{U}^{(2)}$ is a 6×6 matrix. □

Example 6.3. In this example we show that $\mathbf{U}^{(2)}$ having full column rank is not necessary for uniqueness. In Stegeman [37] this was shown for the asymmetric case. The smallest R for which we have found such an example is $R = 7$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}. \tag{6.4}$$

The right-hand side of (5.7) equals $\Phi(4) = 20$. Hence, after deleting redundant rows, $\mathbf{U}^{(2)}$ has 20 rows left. Since it has $R(R-1)/2 = 21$ columns, it cannot have full column rank. Next, we show that condition (4.10) holds, which implies uniqueness of $(\mathbf{A}, \mathbf{A}, \mathbf{I}_7)$. It can be verified that $(\mathbf{A} \odot \mathbf{A})\mathbf{d} = \mathbf{f} \otimes \mathbf{f}$ yields the equations

$$\begin{aligned} d_1 + d_5 + d_6 + d_7 &= f_1^2, & d_5 + 2d_6 + 8d_7 &= f_2 f_3, \\ d_5 + 2d_6 + 4d_7 &= f_1 f_2, & d_5 + 4d_6 + 4d_7 &= f_2 f_4, \\ d_5 + d_6 + 2d_7 &= f_1 f_3, & d_3 + d_5 + d_6 + 4d_7 &= f_3^2, \\ d_5 + 2d_6 + d_7 &= f_1 f_4, & d_5 + 2d_6 + 2d_7 &= f_3 f_4, \\ d_2 + d_5 + 4d_6 + 16d_7 &= f_2^2, & d_4 + d_5 + 4d_6 + d_7 &= f_4^2. \end{aligned}$$

Rewriting $(f_1 f_2)(f_3 f_4) = (f_1 f_3)(f_2 f_4)$ yields $d_5 d_6 = 0$. Rewriting $(f_1 f_2)(f_3 f_4) = (f_1 f_4)(f_2 f_3)$ yields $3d_5 d_7 + 6d_6 d_7 = 0$. Together, this implies that at most one of d_5, d_6, d_7 can be nonzero. Let $d_5 \neq 0$ and $d_6 = d_7 = 0$. The equations above imply that $f_1 = f_2 = f_3 = f_4 \neq 0$, $d_5 = f_1^2$, and $d_1 = d_2 = d_3 = d_4 = 0$. Next, let $d_6 \neq 0$ and $f_5 = f_7 = 0$. Then we obtain $f_1 = f_3$, $f_2 = f_4 = 2f_1$, $d_6 = f_1^2$, and $d_1 = d_2 = d_3 = d_4 = 0$. Analogously, if $d_7 \neq 0$ and $d_5 = d_6 = 0$, then $f_1 = f_4$, $f_2 = 4f_1$, $f_3 = 2f_1$, $d_7 = f_1^2$, and $d_1 = d_2 = d_3 = d_4 = 0$. Next, suppose $d_1 \neq 0$. From the above it follows that $d_5 = d_6 = d_7 = 0$. This implies $d_1 = f_1^2 \neq 0$, $f_2 = f_3 = f_4 = 0$, and $d_2 = d_3 = d_4 = 0$. The cases $d_2 \neq 0$, $d_3 \neq 0$, $d_4 \neq 0$ can be treated in the same way. This shows that $\omega(\mathbf{d}) \leq 1$. Hence, condition (4.10) holds. ∎

7. Discussion. In this paper, we have proven necessary, sufficient, necessary and sufficient, and generic uniqueness conditions for nth order tensor decompositions with some form of symmetry. The analogues for the asymmetric case can be found in Stegeman [38]. When comparing the symmetric and asymmetric cases, the following can be observed. The necessary condition concerning proportional columns (Lemma 3.2) carries over to the symmetric case, although requiring a more complicated proof. The necessary condition $\operatorname{rank}(\bigodot_{i \neq j}^{n} \mathbf{A}^{(i)}) = R$ could be proven for the symmetric case only if mode j is not included in the symmetry. The remaining case is still an open question. If $\operatorname{rank}(\mathbf{A}^{(n)}) = R$ and $\operatorname{rank}(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)}) = R$, then alternative decompositions necessarily feature the same form of symmetry as the original decomposition (see Theorem 4.5). This fact yields natural analogues of the uniqueness conditions in Theorems 4.1 and 4.3 and Corollary 4.4 (see section 4.2). The most striking difference between the symmetric and asymmetric case occurs in the generic uniqueness bounds on $R(R-1)/2$ presented in section 5. In the presence of symmetry, these bounds are much lower than in the asymmetric case.

As mentioned in section 1, decompositions with some form of symmetry mostly occur in signal processing. For complex-valued decompositions the symmetry often takes the form of one component matrix being equal to the complex conjugate of another. Since we consider only the real-valued case, uniqueness of such decompositions is not directly covered by our results. However, analogous to [38], our results can be translated easily to the complex case. To do this, we must keep in mind that our vectors live in a complex vector space $\mathbb{C}^m$ with inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^H \mathbf{x}$ and norm $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, where ^{H} denotes the Hermitian or conjugated transpose. As in $\mathbb{R}^m$, vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal when $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Also, vectors $\mathbf{x}_1, \ldots, \mathbf{x}_q \in \mathbb{C}^m$ are linearly independent when $a_1 \mathbf{x}_1 + \cdots + a_q \mathbf{x}_q = \mathbf{0}$ implies $a_1 = \cdots = a_q = 0$ for scalars $a_1, \ldots, a_q \in \mathbb{C}$. Moreover, the determinant of a complex matrix is defined identical to the determinant of a real matrix, and its relation to the matrix rank is identical. The considerations above imply that, in order to translate our uniqueness proofs to the complex case, we must replace the ordinary transpose ^{T} by ^{H} where orthogonality is involved. However, in cases where the transpose is due to the formulation of the decomposition such as in (2.1), the transpose should not be changed. See also [18], where all uniqueness results (for the

asymmetric decomposition with $n = 3$) are proven for the complex case. A translation of our results to the complex case is as follows. Lemmas 3.3 and 3.4 also hold for the complex case. Note that in (3.4) we don't have to worry about the term under the square root being positive. The results in section 4.2 also hold for the complex case. In Theorem 4.6 we may take $\mathbf{f}_i$ equal to the complex conjugate of $\mathbf{f}_j$ if this holds for $\mathbf{A}^{(i)}$ and $\mathbf{A}^{(j)}$. The results of section 5.2 are obtained by identifying minors with identical terms. This method yields the same results in the complex case. Hence, after translating, the results in this paper can be applied to complex-valued decompositions as well.

Appendix A. Proof of Theorem 5.2. Here, $\underline{\mathbf{Y}}$ is an $I_1 \times I_1$ symmetric matrix $\mathbf{Y}$. The $n - 1 = 2$ matrix unfoldings of $\mathbf{Y}$ are $\mathbf{Y}$ itself and $\mathbf{Y}^T$, which are identical. Hence, we need to consider only the 2×2 minors of $\mathbf{Y}$. We denote the equation of a minor as $y_{ij}y_{pq} = y_{iq}y_{pj}$, where we refer to $y_{ij}y_{pq}$ as its first term and to $y_{iq}y_{pj}$ as its second term. For convenience, we use the notation (i, j, p, q) for the minor as well. Without loss of generality, we assume $i < p$ and $j < q$. This yields $I_1^2(I_1 - 1)^2/4$ minors to start with. In the asymmetric case, all these minors are nonredundant. With symmetry present, we proceed as follows.

As stated in section 5.2, we identify minors with identical terms. Since $\mathbf{Y}$ is symmetric, we have $y_{ij} = y_{ji}$. First, we consider minors with an identical term due to swapping indices in both y of that term. Suppose (i_1, j_1, p_1, q_1) and (i_2, j_2, p_2, q_2) are two such minors. They have identical first terms due to swapping indices in both y if either $(i_1, j_1) = (j_2, i_2)$ and $(p_1, q_1) = (q_2, p_2)$, or $(i_1, j_1) = (q_2, p_2)$ and $(p_1, q_1) = (j_2, i_2)$. In the latter case we have $i_1 = q_2 > j_2 = p_1$ which contradicts $i_1 < p_1$. Hence, we assume the former case holds. Since $(i_1, q_1) = (j_2, p_2)$ and $(p_1, j_1) = (q_2, i_2)$, the symmetry also implies that their second terms are identical. Analogously, identical second terms due to swapping indices in both y implies identical first terms. Next, suppose the first term of (i_1, j_1, p_1, q_1) is identical to the second term of (i_2, j_2, p_2, q_2) due to swapping indices in both y. Again there are two possibilities. Either $(i_1, j_1) = (q_2, i_2)$ and $(p_1, q_1) = (j_2, p_2)$, or $(i_1, j_1) = (j_2, p_2)$ and $(p_1, q_1) = (q_2, i_2)$. In the former case we have $i_1 = q_2 > j_2 = p_1$ which contradicts $i_1 < p_1$. In the latter case we have $j_1 = p_2 > i_2 = q_1$ which contradicts $j_1 < q_1$. Hence, both cases are impossible. Therefore, having identical terms due to swapping indices in both y implies having completely identical minors. These identical minors are excluded by adding the following constraint on (i, j, p, q):

$$(i, p) \preccurlyeq (j, q) \quad \Leftrightarrow \quad \text{either } i < j \quad \text{or} \quad (i = j \,\&\, p \leq q). \tag{A.1}$$

The number of minors we are left with equals

$$\frac{1}{2}\frac{I_1(I_1 - 1)}{2}\left(\frac{I_1(I_1 - 1)}{2} + 1\right). \tag{A.2}$$

Next, we consider minors (i_1, j_1, p_1, q_1) and (i_2, j_2, p_2, q_2) with an identical term due to swapping the indices of only one y in that term. When their first terms are identical, we have four possibilities:

$$(i_1, j_1) = (i_2, j_2) \quad \text{and} \quad (p_1, q_1) = (q_2, p_2), \quad \text{or} \tag{A.3}$$

$$(i_1, j_1) = (j_2, i_2) \quad \text{and} \quad (p_1, q_1) = (p_2, q_2), \quad \text{or} \tag{A.4}$$

$$(i_1, j_1) = (p_2, q_2) \quad \text{and} \quad (p_1, q_1) = (j_2, i_2), \quad \text{or} \tag{A.5}$$

$$(i_1, j_1) = (q_2, p_2) \quad \text{and} \quad (p_1, q_1) = (i_2, j_2). \tag{A.6}$$

In (A.5) we have $i_1 = p_2 > i_2 = q_1$ and $i_1 < p_1 = j_2 < q_2 = j_1 < q_1$ which contradict each other. In (A.6) we have $j_1 = p_2 > i_2 = p_1$ and $j_1 < q_1 = j_2 < q_2 = i_1 < p_1$ which contradict each other. In (A.4) we have $(i_1, p_1) \preccurlyeq (j_1, q_1)$ and $(j_1, p_1) \preccurlyeq (i_1, q_1)$, which implies $i_1 = j_1 = i_2 = j_2$. Hence, we have only one minor instead of two. In (A.3) we have $(i_1, p_1) \preccurlyeq (j_1, q_1)$ and $(i_1, q_1) \preccurlyeq (j_1, p_1)$, which implies $p_1 \neq q_1$ and $i_1 < j_1$ in order to have two minors. Hence, (A.3) is possible for

$$i_1 < j_1 < p_1 < q_1 \quad \text{or} \quad i_1 < j_1 < q_1 < p_1. \tag{A.7}$$

Analogously, it can be shown that the two minors have identical second terms due to swapping indices in one y only if $(i_1, q_1) = (i_2, q_2)$ and $(p_1, j_1) = (j_2, p_2)$, which implies

$$i_1 < j_1 < p_1 < q_1 \quad \text{or} \quad i_1 < p_1 < j_1 < q_1. \tag{A.8}$$

Also, we obtain that the first term of (i_1, j_1, p_1, q_1) is identical to the second term of (i_2, j_2, p_2, q_2) due to swapping indices in one y only if $(i_1, j_1) = (i_2, q_2)$ and $(p_1, q_1) = (j_2, p_2)$, which implies $i_1 < p_1 < j_1 < q_1$. Together with (A.7) and (A.8), this implies that for $i < j < p < q$ the following holds:

- Minor (i, j, p, q) has second term identical to second term of minor (i, p, j, q).
- Minor (i, p, j, q) has first term identical to second term of minor (i, j, q, p).
- Minor (i, j, q, p) has first term identical to first term of minor (i, j, p, q).

Hence, each $i < j < p < q$ identifies three minors of which one is redundant. The number of subsets (i, j, p, q) with $i < j < p < q$ equals $\binom{I_1}{4}$, which is equal to the number of redundant minors due to identical terms by swapping indices of one y only. Therefore, the total number of nonredundant minors equals (A.2) minus $\binom{I_1}{4}$, which is exactly $\mathbf{\Phi}(I_1)$. This completes the proof.

Appendix B. Proof of Theorem 5.3. Here, $\underline{\mathbf{Y}}$ is an $I_1 \times I_1 \times I_3$ tensor with symmetry in the first two modes. We denote the $I_1 \times I_1$ symmetric frontal slices of $\underline{\mathbf{Y}}$ as $\mathbf{Y}_1, \ldots, \mathbf{Y}_{I_3}$. The first and second matrix unfoldings are given by $[\mathbf{Y}_1 | \ldots | \mathbf{Y}_{I_3}]$ and $[\mathbf{Y}_1^T | \ldots | \mathbf{Y}_{I_3}^T]$, respectively. Hence, they are identical. The entries of $\underline{\mathbf{Y}}$ have three indices. A minor of a matrix unfolding of $\underline{\mathbf{Y}}$ corresponds to an equation $y_{\mathcal{A}} y_{\mathcal{D}} = y_{\mathcal{B}} y_{\mathcal{C}}$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ contain three indices. Let $\mathrm{dif}(\mathcal{A}, \mathcal{D})$ denote the number of different indices in $\mathcal{A}$ and $\mathcal{D}$ (i.e., indices with different values at the same position). We have $\mathrm{dif}(\mathcal{A}, \mathcal{D}) \in \{2, 3\}$. We refer to this number as the *order* of the minor. Hence, a minor can have order 2 or 3, where a minor of order 2 has one fixed index. An order 2 minor with fixed third index is a 2×2 minor of a slice $\mathbf{Y}_k$ and corresponds to an equation $y_{ijk} y_{pqk} = y_{iqk} y_{pjk}$. From the proof of Theorem 5.2, it follows that of such minors there are

$$I_3 \mathbf{\Phi}(I_1) \tag{B.1}$$

nonredundant. Indeed, there are I_3 symmetric frontal slices, and each slice has $\mathbf{\Phi}(I_1)$ nonredundant minors. By symmetry, the minors of order 2 with fixed first index are identical to the minors of order 2 with fixed second index. These correspond to $y_{ijk} y_{iqr} = y_{ijr} y_{iqk}$, where we set $j < q$ and $k < r$ without loss of generality. Since the third indices k and r are distinct, these minors do not share terms with the order 2 minors having fixed third index. It follows that there are

$$I_1 \frac{I_1(I_1-1)}{2} \frac{I_3(I_3-1)}{2} \tag{B.2}$$

nonredundant minors of order 2 with first (or second) index fixed. Next, we consider minors of order 3. As observed in [38], each such minor corresponds to a $2 \times 2 \times 2$ subtensor of $\underline{\mathbf{Y}}$ with frontal slices

$$\begin{bmatrix} y_{ijk} & y_{iqk} \\ y_{pjk} & y_{pqk} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_{ijr} & y_{iqr} \\ y_{pjr} & y_{pqr} \end{bmatrix}, \tag{B.3}$$

where we set $i < p$, $j < q$, $k < r$ without loss of generality. By symmetry, swapping i and j, and p and q, yields an identical subtensor. Hence, we add the constraint $(i, p) \preccurlyeq (j, q)$. Analogous to (A.2), this yields a total number of subtensors equal to

$$\frac{I_3(I_3-1)}{2} \frac{I_1(I_1-1)}{4} \left(\frac{I_1(I_1-1)}{2} + 1 \right). \tag{B.4}$$

It is shown in [38, section 5.2] that, in absence of symmetry, each subtensor corresponds to six distinct minors of order 3 that equate four distinct terms. Only three of the six minors are nonredundant. These can be written as

$$y_{ijk}y_{pqr} = y_{pjk}y_{iqr} = y_{iqk}y_{pjr} = y_{ijr}y_{pqk}. \tag{B.5}$$

In the asymmetric case, [38] shows that minors corresponding to different subtensors do not have identical terms. Moreover, minors of different orders do not have identical terms. In the current symmetric case, this is not true.

A subtensor (i, j, k, p, q, r) with i, j, p, q distinct does not share terms with a minor of order 2, since the terms do not have fixed indices. Next, suppose i, j, p, q are not distinct. We have the following cases. If $i = j$ and $p = q$, then $y_{pjk}y_{iqr} = y_{iqk}y_{pjr}$ in (B.5) and the subtensor has two nonredundant minors instead of three. If $i = j$ and $p < q$, then $y_{pjk}y_{iqr} = y_{ipk}y_{iqr}$ is equal to $y_{ipr}y_{iqk} = y_{iqk}y_{pjr}$ by an order 2 minor having fixed first index. Hence, here also the subtensor has two nonredundant minors. Analogously, if $i < j$ and $p = q$, then $y_{pjk}y_{iqr} = y_{pjk}y_{pir}$ is equal to $y_{pjr}y_{pik} = y_{iqk}y_{pjr}$ by an order 2 minor. Finally, if $i < j = p < q$, then $y_{ijk}y_{pqr} = y_{jik}y_{jqr}$ is equal to $y_{jir}y_{jqk} = y_{ijr}y_{pqk}$ by an order 2 minor. We conclude that in all cases where i, j, p, q are not distinct the subtensor (i, j, k, p, q, r) has two nonredundant minors.

Next, we consider two subtensors, denoted by $(i_1, j_1, k_1, p_1, q_1, r_1)$ and $(i_2, j_2, k_2, p_2, q_2, r_2)$, and determine which of the four terms of subtensor 1 can be identical to a term of subtensor 2 by swapping the first two indices in y. First, we consider identical terms due to swapping indices of two y in a term. It can be shown that this is not possible. In each case, a contradiction with $i_1 < p_1$, $j_1 < q_1$, $k_1 < r_1$ is obtained, or the two subtensors are identical. The proof of this is in the same way as the proof of Theorem 5.2, and it is omitted. Furthermore, it can be shown (proof omitted) that if the subtensors have identical terms due to swapping indices of one y in a term, then i_1, j_1, p_1, q_1 are distinct (and i_2, j_2, p_2, q_2 as well).

Let i, j, p, q be distinct with $i < j < p < q$. Also, let $k < r$. It can be shown that identical terms occur only within groups of three subtensors (i, j, k, p, q, r), (i, j, k, q, p, r), and (i, p, k, j, q, r). We refer to the four terms in (B.5) as the terms 1, 2, 3, 4 in order of appearance. As in the proof of Theorem 5.2, the following hold:

- Subtensor (i, j, k, p, q, r) has terms 1 and 4 identical to terms 1 and 4 of subtensor (i, j, k, q, p, r), respectively.

- Subtensor (i, j, k, q, p, r) has terms 2 and 3 identical to terms 4 and 1 of subtensor (i, p, k, j, q, r), respectively.
- Subtensor (i, p, k, j, q, r) has terms 2 and 3 identical to terms 2 and 3 of subtensor (i, j, k, p, q, r), respectively.

It follows that the three subtensors together (which have twelve terms in total) have six distinct terms that should all be equal. Five minors are enough for this, and these are nonredundant. The total number of groups of three subtensors as above equals $(I_3(I_3-1)/2)\binom{I_1}{4}$. Counting two nonredundant minors per subtensor, the total number of nonredundant minors of order 3 equals two times (B.4) minus $(I_3(I_3-1)/2)\binom{I_1}{4}$, which can be rewritten as

$$\frac{I_3(I_3-1)}{2}\left(2\Phi(I_1)+\binom{I_1}{4}\right)=\frac{I_3(I_3-1)}{2}\Psi(I_1). \tag{B.6}$$

Adding (B.1), (B.2), and (B.6) yields the right-hand side of (5.8). This completes the proof.

Appendix C. Proof of Theorem 5.4. Since this proof is a fairly straightforward generalization of the proof of Theorem 5.3, we will be briefer in its presentation. The tensor $\underline{\mathbf{Y}}$ has size $I_1 \times I_1 \times I_3 \times I_4$ and is symmetric in the first two modes. A minor of a matrix unfolding of $\underline{\mathbf{Y}}$ corresponds to an equation $y_{\mathcal{A}}y_{\mathcal{D}} = y_{\mathcal{B}}y_{\mathcal{C}}$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ contain four indices. The order of a minor is equal to $\mathrm{dif}(\mathcal{A}, \mathcal{D}) \in \{2, 3, 4\}$. Analogous to (B.1), the number of nonredundant minors of order 2 with fixed third and fourth indices equals

$$I_3 I_4 \Phi(I_1). \tag{C.1}$$

Analogous to (B.2), the number of nonredundant minors of order 2 with fixed first (or second) and third indices equals

$$I_1 I_3 \frac{I_1(I_1-1)}{2}\frac{I_4(I_4-1)}{2}. \tag{C.2}$$

The number of nonredundant minors of order 2 with fixed first (or second) and fourth indices equals

$$I_1 I_4 \frac{I_1(I_1-1)}{2}\frac{I_3(I_3-1)}{2}. \tag{C.3}$$

For a minor of order 2 with fixed first and second indices, the latter can be swapped to obtain an identical minor. The number of unique pairs of first and second indices equals $I_1(I_1+1)/2$. Hence, the number of nonredundant minors of this type equals

$$\frac{I_1(I_1+1)}{2}\frac{I_3(I_3-1)}{2}\frac{I_4(I_4-1)}{2}. \tag{C.4}$$

Next, we consider minors of order 3. These have one fixed index and define $2 \times 2 \times 2$ subtensors of $\underline{\mathbf{Y}}$ analogous to (B.3). Analogous to (B.6), the number of nonredundant minors of order 3 with fixed third or fourth index equals

$$I_3\frac{I_4(I_4-1)}{2}\Psi(I_1)+I_4\frac{I_3(I_3-1)}{2}\Psi(I_1). \tag{C.5}$$

Of the minors of order 3 with first or second index fixed, we need only consider those with first index fixed. Analogous to (B.5), the three minors corresponding to a $2 \times 2 \times 2$ subtensor of $\underline{\mathbf{Y}}$ can be written as

$$y_{ijkl}y_{iqrs} = y_{iqkl}y_{ijrs} = y_{ijrl}y_{iqks} = y_{ijks}y_{iqrl}, \tag{C.6}$$

where $j < q$, $k < r$, $l < s$. Since there is no symmetry in the second, third, and fourth indices, all three minors of a subtensor are nonredundant. This yields a number of nonredundant minors of order 3 with first (or second) index fixed equal to

$$3I_1 \frac{I_1(I_1-1)}{2} \frac{I_3(I_3-1)}{2} \frac{I_4(I_4-1)}{2}. \tag{C.7}$$

Finally, we consider minors of order 4. As observed in [38], each such minor corresponds to a $2 \times 2 \times 2 \times 2$ subtensor of $\underline{\mathbf{Y}}$ with frontal $2 \times 2 \times 2$ tensors

$$\left[\begin{array}{cc|cc} y_{ijkl} & y_{iqkl} & y_{ijrl} & y_{iqrl} \\ y_{pjkl} & y_{pqkl} & y_{pjrl} & y_{pqrl} \end{array}\right] \text{ and } \left[\begin{array}{cc|cc} y_{ijks} & y_{iqks} & y_{ijrs} & y_{iqrs} \\ y_{pjks} & y_{pqks} & y_{pjrs} & y_{pqrs} \end{array}\right], \tag{C.8}$$

where we set $i < p$, $j < q$, $k < r$, $l < s$ without loss of generality. As in the proof of Theorem 5.3, we add the constraint $(i, p) \preccurlyeq (j, q)$. Analogous to (B.4), this yields a total number of subtensors equal to

$$\frac{I_3(I_3-1)}{2} \frac{I_4(I_4-1)}{2} \frac{I_1(I_1-1)}{4} \left(\frac{I_1(I_1-1)}{2} + 1 \right). \tag{C.9}$$

It is shown in [38, section 5.2] that, in absence of symmetry, each subtensor corresponds to 16 distinct minors of order 4 that equate 8 distinct terms. Only 7 of the 16 minors are nonredundant. These can be written as

$$\begin{aligned} y_{ijkl}y_{pqrs} &= y_{ijks}y_{pqrl} = y_{pjkl}y_{iqrs} = y_{pjks}y_{iqrl} \\ &= y_{iqkl}y_{pjrs} = y_{iqks}y_{pjrl} = y_{ijrl}y_{pqks} = y_{ijrs}y_{pqkl}. \end{aligned} \tag{C.10}$$

Note that the form of the 8 terms is analogous to (B.5). Analogous to the proof of Theorem 5.3, it can be shown that if $i = j$ and $p = q$, or if $i = j$ and $p < q$, or if $i < j$ and $p = q$, or if $i < j = p < q$, then the subtensor has only four nonredundant minors. When i, j, p, q are distinct, identical terms occur only between subtensors (i, j, k, l, p, q, r, s), (i, j, k, l, q, p, r, s), and (i, p, k, l, j, q, r, s). Analogous to the proof of Theorem 5.3, the following hold:

- Subtensor (i, j, k, l, p, q, r, s) has terms 1, 2, 7, 8 identical to terms 1, 2, 7, 8 of subtensor (i, j, k, l, q, p, r, s), respectively.
- Subtensor (i, j, k, l, q, p, r, s) has terms 3, 4, 5, 6 identical to terms 8, 7, 1, 2 of subtensor (i, p, k, l, j, q, r, s), respectively.
- Subtensor (i, p, k, l, j, q, r, s) has terms 3, 4, 5, 6 identical to terms 3, 4, 5, 6 of subtensor (i, j, k, l, p, q, r, s), respectively.

It follows that the three subtensors together (which have 24 terms in total) have 12 distinct terms that should all be equal. Only 11 minors are enough for this, and these are nonredundant. The total number of groups of three subtensors as above equals $(I_3(I_3-1)/2)(I_4(I_4-1)/2)\binom{I_1}{4}$. Counting four nonredundant minors per subtensor, the total number of nonredundant minors of order 4 equals four times (C.9) minus

$(I_3(I_3-1)/2)(I_4(I_4-1)/2)\binom{I_1}{4}$, which can be rewritten as

$$\frac{I_3(I_3-1)}{2}\frac{I_4(I_4-1)}{2}\left(4\Phi(I_1)+3\binom{I_1}{4}\right)=\frac{I_3(I_3-1)}{2}\frac{I_4(I_4-1)}{2}\boldsymbol{\Omega}(I_1). \tag{C.11}$$

Adding (C.1)–(C.5), (C.7), and (C.11) yields the right-hand side of (5.9). This completes the proof.

Appendix D. Proof of Theorem 5.5. Here, we consider the special case of Theorem 5.4 in which $\mathbf{A}^{(3)}=\mathbf{A}^{(4)}$. The tensor $\underline{\mathbf{Y}}$ has size $I_1\times I_1\times I_3\times I_3$ and is symmetric in the first two modes and in the last two modes. The numbers of nonredundant minors of orders 2 and 3 follow from the proof of Theorem 5.4. First, we consider minors of order 2, which have two indices fixed. Suppose the first and second index are fixed. By symmetry, there are $I_1(I_1+1)/2$ unique pairs of the first two indices. Analogous to (C.1), we obtain $(I_1(I_1+1)/2)\boldsymbol{\Phi}(I_3)$ nonredundant minors of this type. For the minors with fixed third and fourth indices, we can simply swap I_1 and I_3 to obtain the number of nonredundant minors. The number of nonredundant minors with fixed first (or second) and third (or fourth) indices is analogous to (C.2) and (C.3). For the total number of nonredundant minors of order 2, we obtain

$$\frac{I_1(I_1+1)}{2}\boldsymbol{\Phi}(I_3)+\frac{I_3(I_3+1)}{2}\boldsymbol{\Phi}(I_1)+I_1I_3\frac{I_1(I_1-1)}{2}\frac{I_3(I_3-1)}{2}. \tag{D.1}$$

Next, we consider minors of order 3, which have one index fixed. Analogous to (C.5), the number of nonredundant minors with first (or second) index fixed plus the number of nonredundant minors with third (or fourth) index fixed equals

$$I_1\frac{I_1(I_1-1)}{2}\boldsymbol{\Psi}(I_3)+I_3\frac{I_3(I_3-1)}{2}\boldsymbol{\Psi}(I_1). \tag{D.2}$$

Finally, we consider minors of order 4, which correspond to $2\times2\times2\times2$ subtensors as in (C.8). Since we have symmetry in the first two and the last two modes, we require not only $(i,p)\preccurlyeq(j,q)$ but also $(k,r)\preccurlyeq(l,s)$. This yields a total number of subtensors equal to

$$\left(\frac{I_1(I_1-1)}{4}\left(\frac{I_1(I_1-1)}{2}+1\right)\right)\left(\frac{I_3(I_3-1)}{4}\left(\frac{I_3(I_3-1)}{2}+1\right)\right). \tag{D.3}$$

It can be shown that if i,j,p,q are not distinct and also k,l,r,s are not distinct, then the subtensor has only two nonredundant minors. If i,j,p,q are not distinct, but k,l,r,s are distinct, then the three subtensors (i,j,k,l,p,q,r,s), (i,j,k,l,p,q,s,r), (i,j,k,r,p,q,l,s) together have six nonredundant minors. Analogously, if i,j,p,q are distinct, but k,l,r,s are not, then the three subtensors (i,j,k,l,p,q,r,s), (i,j,k,l,q,p,r,s), (i,p,k,l,j,q,r,s) together have six nonredundant minors. Hence, in these cases there are two nonredundant minors per subtensor.

If $i<j<p<q$ and $k<l<r<s$, then identical terms occur only within a group of nine subtensors: (i,j,k,l,p,q,r,s), (i,j,k,l,q,p,r,s), (i,p,k,l,j,q,r,s), (i,j,k,l,p,q,s,r), (i,j,k,l,q,p,s,r), (i,p,k,l,j,q,s,r), (i,j,k,r,p,q,l,s), (i,j,k,r,q,p,l,s), and (i,p,k,r,j,q,l,s). It can be shown that these 9 subtensors together have 18 distinct terms that should all be equal. Only 17 minors are enough for this, and these are nonredundant. The total number of groups of 9 subtensors as above equals $\binom{I_1}{4}\binom{I_3}{4}$. Count-

ing two nonredundant minors per subtensor, the total number of nonredundant minors of order 4 equals two times (D.3) minus $\binom{I_1}{4}\binom{I_3}{4}$, which can be rewritten as

$$2\left(\Phi(I_1)+\binom{I_1}{4}\right)\left(\Phi(I_3)+\binom{I_3}{4}\right)-\binom{I_1}{4}\binom{I_3}{4}=\Delta(I_1,I_3). \tag{D.4}$$

Adding (D.1), (D.2), and (D.4) yields the right-hand side of (5.10). This completes the proof.

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